Homework 2 - Sketch of Solutions

1.5 **#2a** $x\begin{pmatrix}1 & -3\\ -2 & 4\end{pmatrix} + y\begin{pmatrix}-2 & 6\\ 4 & -8\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

$\begin{pmatrix}x-2y & -3x+6y\\ -2x+4y & 4x-8y\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ $\therefore x = 2y$

Set $y=1$ so $x=2$. $\therefore$ linear dependent

#5 $a_0 + a_1 x + \cdots + a_n x^n = 0$. But a polynomial $= 0 \Leftrightarrow$ all coefficients $= 0$ $\therefore a_i = 0$ all i

#9 Suppose $\{u, v\}$ linearly dep. $au + bv = 0$ a, b not both 0 say $a \neq 0$ $\therefore u = (a^{-1}(-b))v$ $\therefore u$ is a scalar multiple of v. Similarly if $b \neq 0$

Conversely suppose $u = av$ some a

$\therefore u + (-a)v = 0$ linear dep. Similarly if $v = bu$

#17 $M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & & a_{nn}\end{pmatrix}$ Call the columns $M_1, M_2, \ldots, M_n$

Suppose $b_1 M_1 + \cdots + b_n M_n = 0$

$$\begin{pmatrix} b_1 a_{11} + b_2 a_{12} + \cdots + b_n a_{1n}\\ b_2 a_{22} + \cdots + b_n a_{2n}\\ \vdots\\ b_{n-1} a_{n-1\,n-1} + b_n a_{n-1\,n}\\ b_n a_{nn}\end{pmatrix} = \begin{pmatrix}0\\0\\ \vdots\\0\\0\end{pmatrix}$$

$b_n a_{nn} = 0$ $\therefore b_n = 0$ $b_{n-1} a_{n-1\,n-1} = 0$ $\therefore b_{n-1} = 0$

$\ldots$ $b_1 = 0$

#18 Let $p_1(x), \ldots, p_k(x)$ be a finite set of polynomials with degrees $n_1, n_2, \ldots, n_k$ with $n_1 > n_2 > \cdots > n_k$

Suppose $a_1 p_1(x) + \cdots + a_k p_k(x) = 0$

Then the highest degree on the LHS is m_1

and we have

$$a_1 b_{m_1} x^{m_1} + \text{terms of degree} < m_1 = 0$$

where $p_1(x) = b_{m_1} x^{m_1} + \cdots + b_0$. $\therefore a_1 b_{m_1} = 0$ $\therefore a_1 = 0$

Now consider $a_2 p_2(x) + \cdots + a_k p_k(x) = 0$ and

proceed as above.

1.6 **#3e** Test for linear indep.:

$u(1+2x-x^2) + v(4-2x+x^2) + w(-1+18x-9x^2) = 0$

$u + 4v - w = 0$

$2u - 2v + 18w = 0$ solution $u = -7w$, $v = 2w$ any w

$-u + v - 9w = 0$ take $w = 1$ for example : $u = -7$, $v = 1$, $w = 1$

The vectors are linearly indep. $\therefore$ a basis (why?)

#4 No $\dim P_3(\mathbb{R}) = 4$ and would require at least 4

polynomial for spanning.

#7 $u_3 = -4u_1$ so delete u_3. Start with u_1, u_2 not a

multiple of u_1 $\therefore u_1, u_2$ linearly indep. Is u_4 a linear

combination of u_1, u_2? Do there exist x, y

$(1, 37, -17) = x(2, -3, 1) + y(1, 4, -2)$

$$\left.\begin{aligned} 2x + y &= 1 \\ -3x + 4y &= 37 \\ x - 2y &= -17 \end{aligned}\right\} \text{ solution } x = -3,\ y = 7$$

Delete u_4 Have (u_1, u_2, u_5) Show linear indep.

#11 Suffices to show $\{u+v, au\}$ linearly indep.

$x(u+v) + y(au) = 0$

Solution $x = 0$, $y = 0$

#12 Suffice to show $\{u+v+w, v+w, w\}$ linearly indep.

$x(u+v+w) + y(v+w) + zw = 0$ implies $x = y = z = 0$

13 $x_1 - 2x_2 + x_3 = 0$

$2x_1 - 3x_2 + x_3 = 0$ Solution $x_1 = x_3$, $x_2 = x_3$

All solutions $\{(x_3, x_3, x_3) \mid x_3 \in \mathbb{R}\}$ Basis $(1,1,1)$,

more generally (a,a,a) for any $a \neq 0$

16 E_{ij} $i \geq j$ with one in ij position, 0 elsewhere

Count number of E_{ij}

$$n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}$$

17 Let $u_1 \in S$ be non-zero $\exists u_2 \in S$, $u_2 \notin \langle u_1 \rangle$ (otherwise $\{u_1\}$ would span V and so $\dim V$ would $= 1$)

$\therefore \{u_1, u_2\}$ linearly indep. $\exists u_3 \in S$ $u_3 \notin \langle u_1, u_2 \rangle$ (otherwise $\{u_1, u_2\}$ spans V and $\dim V = 2$). ... Continue until have $u_1, \ldots, u_n$ all in S and linearly indep. This is a basis

23 $W_1 = \langle v_1, \ldots, v_k \rangle$, $W_2 = \langle v_1, \ldots, v_k, v \rangle$ Note $W_1 \subseteq W_2$

If v is a linear comb of $v_1, \ldots, v_k$, $W_1 = W_2$ so $\dim W_1 = \dim W_2$

Conversely if $\dim W_1 = \dim W_2$, then $W_1 = W_2$ (Thm 1.11) and v is a linear comb of $v_1, \ldots, v_k$

29 (a) Show $S = \{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_p\}$ basis for $W_1 + W_2$. Clearly S spans $W_1 + W_2$. Show S linearly indep (sketched): Suppose

$a_1 u_1 + \cdots + a_k u_k + b_1 v_1 + \cdots + b_m v_m + c_1 w_1 + \cdots + c_p w_p = 0$

$\therefore a_1 u_1 + \cdots + b_m v_m = (-c_1) w_1 + \cdots + (-c_p) w_p \in W_2$

$\therefore$ LHS $\in W_1 \cap W_2$ $\therefore a_1 u_1 + \cdots + b_m v_m = d_1 u_1 + \cdots + d_k u_k$ some d_i

$\therefore b_1 = \cdots = b_m = 0$ Similarly $c_1 = \cdots = c_p = 0$ $\therefore a_1 = \cdots = a_k = 0$

(b) If $V = W_1 \oplus W_2$, $W_1 \cap W_2 = \{0\}$ so $\dim W_1 \cap W_2 = 0$

$\therefore \dim V = \dim W_1 + \dim W_2$ Conversely suppose $\dim V = \dim W_1 + \dim W_2$ But $\dim V = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$

$\therefore \dim W_1 \cap W_2 = 0$ $\therefore W_1 \cap W_2 = \{0\}$. $\therefore V = W_1 \oplus W_2$